

Lagrangian Dynamic Formulation of a Four-Bar Mechanism with Minimal Coordinates

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March 2006 (Last Corrected: February 2, 2010)

Abstract

This article describes the detailed steps in formulating the dynamic equation of a four-bar mechanism in the minimal coordinate form using Lagrangian formulation. It is well-known that a four-bar mechanism possesses one degree-of-freedom (d.o.f.), and, hence, ideally a single dynamic equation is sufficient to completely describe the dynamics of the closed system. However, using only single coordinate to parameterize the entire system when constructing the dynamic equation tends to be a very tedious process, hence extra number of generalized coordinates are often introduced in order to simplify the modeling process. In this article, we show the systematic process of using the extra number of generalized coordinates to parameterize the configuration of the closed system, and reducing the required coordinates to only one when formulating the dynamic equations in this case.

1 Introduction

We first define the notations and configurations of the four-bar mechanism under consideration. Referring to Figure 1, the four-bar mechanism consists of three moving links (input, coupler, and output links) of lengths l_1 , l_2 , and l_3 , respectively, whose orientation with respect to the horizontal are denoted by the absolute angles of θ , α and ϕ . The ground, measuring from point O to O' , has the length of l_0 . For $i = 1, 2, 3$, the masses of each moving link l_i is m_i , and the moment of inertia of the moving links about the axis through the center of the mass and perpendicular to the plane of its motion is I_i . The mass centers of each link are situated at a distance l_{ci} from the proximal joint of each link.

The purpose is to derive the Lagrangian of the system by first using all the generalized coordinates (θ , α and ϕ) and eliminate the surplus variables (α and ϕ) during the process of position and velocity analysis. By doing so, the Lagrangian can then be completely expressed in terms of θ only, and hence the equation of motion can be completely parameterized by this (single) minimal coordinate.

2 Kinematic analysis

We here present the position and velocity analysis of the four-bar mechanism.

2.1 Position analysis

In position analysis, we begin by writing down the loop-closure equation of the four-bar mechanism as:

$$-l_1 \cos \theta - l_2 \cos \alpha + l_0 + l_3 \cos \phi = 0 \quad (1)$$

$$-l_1 \sin \theta - l_2 \sin \alpha + l_3 \sin \phi = 0 \quad (2)$$

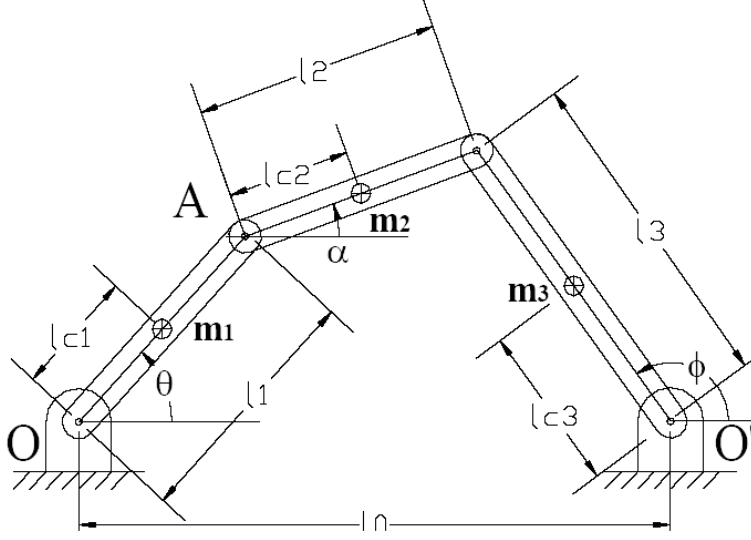


Figure 1: Planar four-bar mechanism under consideration

where Eq. (1) and Eq. (2) are the loop closure constraints in the x and y coordinates, respectively. In this article, we express α and ϕ in terms of θ , so first rearrange the above equations into the following form to eliminate α :

$$l_2 \cos \alpha = l_0 - l_1 \cos \theta + l_3 \cos \phi \quad (3)$$

$$l_2 \sin \alpha = -l_1 \sin \theta + l_3 \sin \phi \quad (4)$$

The sum of the squares of Eq. (3) and Eq. (4) yields:

$$k_1(\theta) \sin \phi + k_2(\theta) \cos \phi + k_3(\theta) = 0 \quad (5)$$

where, k_i ($i = 1, 2, 3$) are functions of θ , and

$$k_1(\theta) = -2l_1 l_3 \sin \theta$$

$$k_2(\theta) = 2l_3(l_0 - l_1 \cos \theta)$$

$$k_3(\theta) = l_0^2 + l_1^2 - l_2^2 + l_3^2 - 2l_0 l_1 \cos \theta$$

Eq. (5) is the well-known *Freudenstein equation*, which can be solved in closed form. This allows us to determine ϕ in terms of θ . To solve Eq. (5), define:

$$t = \tan \frac{\phi}{2} \quad (6)$$

and correspondingly,

$$\sin \phi = \frac{2t}{1+t^2} \quad (7)$$

$$\cos \phi = \frac{1-t^2}{1+t^2} \quad (8)$$

Substitute the above into Eq.(5) yields a quadratic equation in terms of t of:

$$(k_3 - k_2)t^2 + (2k_1)t + (k_3 + k_2) = 0 \quad (9)$$

We then arrive at the solution of:

$$t = \frac{-k_1 \pm \sqrt{k_1^2 + k_2^2 - k_3^2}}{k_3 - k_2} \quad (10)$$

Substituting Eq.(6) into Eq.(10), yields:

$$\phi(\theta) = 2 \cdot \arctan 2(-k_1 \pm \sqrt{k_1^2 + k_2^2 - k_3^2}, k_3 - k_2) \quad (11)$$

Dividing Eq.(4) by Eq.(3), we can then arrive at:

$$\alpha(\theta, \phi) = \arctan 2(-l_1 \sin \theta + l_3 \sin \phi, l_0 - l_1 \cos \theta + l_3 \cos \phi) \quad (12)$$

At this stage, we express α and ϕ completely in terms of θ .

2.2 Velocity analysis

In velocity analysis, differentiating the loop-closure constraints in Eqs. (1) and (2) with respect to time and expressing them in matrix form yield:

$$\underbrace{\begin{bmatrix} l_1 \sin \theta & l_2 \sin \alpha & -l_3 \sin \phi \\ -l_1 \cos \theta & -l_2 \cos \alpha & l_3 \cos \phi \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (13)$$

Since we choose θ as the independent variable, we arrange Eq. (13) into the following:

$$\begin{bmatrix} l_2 \sin \alpha & -l_3 \sin \phi \\ -l_2 \cos \alpha & l_3 \cos \phi \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -l_1 \sin \theta \\ l_1 \cos \theta \end{bmatrix} \dot{\theta} \quad (14)$$

and obtain:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} S_1(\theta, \alpha, \phi) \\ S_2(\theta, \alpha, \phi) \end{bmatrix} \dot{\theta} \quad (15)$$

where $\mathbf{S} = \begin{bmatrix} S_1(\theta, \alpha, \phi) \\ S_2(\theta, \alpha, \phi) \end{bmatrix}$ is the nullspace of matrix \mathbf{A} and

$$S_1(\theta, \alpha, \phi) = \frac{\partial \alpha}{\partial \theta} = \frac{l_1 \sin(\phi - \theta)}{l_2 \sin(\alpha - \phi)} \quad (16)$$

$$S_2(\theta, \alpha, \phi) = \frac{\partial \phi}{\partial \theta} = \frac{l_1 \sin(\alpha - \theta)}{l_3 \sin(\alpha - \phi)} \quad (17)$$

Equation (15) can then be used to determine the velocities $\dot{\alpha}$ and $\dot{\phi}$ in terms of $\dot{\theta}$. The operator \mathbf{S} acts like a filter to maintain the feasible velocities of $\dot{\alpha}$ and $\dot{\phi}$ such that the constraint in Eq. (13) is not violated.

3 Lagrangian formulation

In Lagrangian formulation, the Lagrangian of the entire system is defined by the total kinetic energy minus the total potential energy. We first determine the total kinetic energy of the system as:

$$T = \underbrace{\frac{1}{2} (m_1 \|v_{c1}\|^2 + I_1 \dot{\theta}^2)}_{T_1} + \underbrace{\frac{1}{2} (m_2 \|v_{c2}\|^2 + I_2 \dot{\alpha}^2)}_{T_2} + \underbrace{\frac{1}{2} (m_3 \|v_{c3}\|^2 + I_3 \dot{\phi}^2)}_{T_3} \quad (18)$$

where

$$\begin{aligned} \|v_{c1}\|^2 &= l_{c1}^2 \dot{\theta}^2 \\ \|v_{c2}\|^2 &= l_1^2 \dot{\theta}^2 + l_{c2}^2 \dot{\alpha}^2 + 2l_1 l_{c2} \cos(\theta - \alpha) \dot{\theta} \dot{\alpha} \\ \|v_{c3}\|^2 &= l_{c3}^2 \dot{\phi}^2 \end{aligned}$$

and T_i corresponding to the kinetic energy of link i . We then determine the potential energy of the system as:

$$V = \underbrace{m_1 g y_{c1}}_{V_1} + \underbrace{m_2 g y_{c2}}_{V_2} + \underbrace{m_3 g y_{c3}}_{V_3} \quad (19)$$

where g is the gravitational acceleration, and

$$\begin{aligned} y_{c1} &= l_{c1} \sin \theta \\ y_{c2} &= l_1 \sin \theta + l_{c2} \sin \alpha \\ y_{c3} &= l_{c3} \sin \phi \end{aligned}$$

Finally, the Lagrangian of the entire system is given by:

$$\mathcal{L} = T - V \quad (20)$$

After some mathematical manipulation, we obtain the following Lagrangian expression:

$$\mathcal{L}(\theta, \alpha, \phi, \dot{\theta}, \dot{\alpha}, \dot{\phi}) = J_1 \dot{\theta}^2 + J_2 \dot{\alpha}^2 + J_3 \dot{\phi}^2 + P_1 C_1(\theta, \alpha) \dot{\theta} \dot{\alpha} + G(\theta, \alpha, \phi) \quad (21)$$

where

$$\begin{aligned} J_1 &= \frac{1}{2}(m_1 l_{c1}^2 + I_1 + m_2 l_1^2) \\ J_2 &= \frac{1}{2}(m_2 l_{c2}^2 + I_2) \\ J_3 &= \frac{1}{2}(m_3 l_{c3}^2 + I_3) \\ P_1 &= m_2 l_1 l_{c2} \\ C_1(\theta, \alpha) &= \cos(\theta - \alpha) \\ G(\theta, \alpha, \phi) &= (-m_1 g l_{c1} - m_2 g l_1) \sin \theta - m_2 g l_{c2} \sin \alpha - m_3 g l_{c3} \sin \phi \end{aligned}$$

At this point, we eliminate the velocity terms of $\dot{\alpha}$ and $\dot{\phi}$ from Eq. (21) by using the linear relationship in Eq. (15), and get:

$$\mathcal{L}(\theta, \alpha, \phi, \dot{\theta}, \dot{\alpha}, \dot{\phi}) = [J_1 + J_2 S_1^2(\theta, \alpha, \phi) + J_3 S_2^2(\theta, \alpha, \phi) + P_1 C_1(\theta, \alpha) S_1(\theta, \alpha, \phi)] \dot{\theta}^2 + G(\theta, \alpha, \phi) \quad (22)$$

Although α and ϕ are still remain in the equation, they can be expressed in terms of θ and the Lagrangian is considered completely written in terms of θ .

4 Equation of motion

After constructing the expression of Lagrangian, we can determine the equation of motion of the entire system by:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = \tau_{ext} \quad (23)$$

We first determine:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2 \left[J_1 + J_2 S_1^2(\theta, \alpha, \phi) + J_3 S_2^2(\theta, \alpha, \phi) + P_1 C_1(\theta, \alpha) S_1(\theta, \alpha, \phi) \right] \dot{\theta} \quad (24)$$

Then,

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) &= 2 \left[J_1 + J_2 S_1^2(\theta, \alpha, \phi) + J_3 S_2^2(\theta, \alpha, \phi) + P_1 C_1(\theta, \alpha) S_1(\theta, \alpha, \phi) \right] \ddot{\theta} \\
&+ 2 \left[2J_2 S_1(\theta, \alpha, \phi) \left(\frac{\partial S_1(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \right) + \right. \\
&2J_3 S_2(\theta, \alpha, \phi) \left(\frac{\partial S_2(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial S_2(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial S_2(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \right) + \\
&P_1 \left(C_1(\theta, \alpha) \left(\frac{\partial S_1(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \right) \right. \\
&\left. \left. + S_1(\theta, \alpha, \phi) \left(\frac{\partial C_1(\theta, \alpha)}{\partial \theta} + \frac{\partial C_1(\theta, \alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} \right) \right) \right] \dot{\theta}^2 \quad (25)
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= \left[2J_2 S_1(\theta, \alpha, \phi) \left(\frac{\partial S_1(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \right) + \right. \\
&2J_3 S_2(\theta, \alpha, \phi) \left(\frac{\partial S_2(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial S_2(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial S_2(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \right) + \\
&P_1 \left(C_1(\theta, \alpha) \left(\frac{\partial S_1(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial S_1(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \right) \right. \\
&\left. \left. + S_1(\theta, \alpha, \phi) \left(\frac{\partial C_1(\theta, \alpha)}{\partial \theta} + \frac{\partial C_1(\theta, \alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} \right) \right) \right] \dot{\theta}^2 \\
&+ \frac{\partial G(\theta, \alpha, \phi)}{\partial \theta} + \frac{\partial G(\theta, \alpha, \phi)}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial G(\theta, \alpha, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \theta} \quad (26)
\end{aligned}$$

The full dynamic equation can then be written as:

$$\begin{aligned}
&2 \left[J_1 + J_2 S_1^2 + J_3 S_2^2 + P_1 C_1 S_1 \right] \ddot{\theta} \\
&+ \left[2J_2 S_1 \left(\frac{\partial S_1}{\partial \theta} + S_1 \frac{\partial S_1}{\partial \alpha} + S_2 \frac{\partial S_1}{\partial \phi} \right) + 2J_3 S_2 \left(\frac{\partial S_2}{\partial \theta} + S_1 \frac{\partial S_2}{\partial \alpha} + S_2 \frac{\partial S_2}{\partial \phi} \right) \right. \\
&\left. + P_1 \left(C_1 \left(\frac{\partial S_1}{\partial \theta} + S_1 \frac{\partial S_1}{\partial \alpha} + S_2 \frac{\partial S_1}{\partial \phi} \right) + S_1 \left(\frac{\partial C_1}{\partial \theta} + S_1 \frac{\partial C_1}{\partial \alpha} \right) \right) \right] \dot{\theta} \\
&\quad - \frac{\partial G}{\partial \theta} - S_1 \frac{\partial G}{\partial \alpha} - S_2 \frac{\partial G}{\partial \phi} = \tau_\theta \quad (27)
\end{aligned}$$

Or, in a compact form, it can be written as:

$$M(\theta) \ddot{\theta} + V(\theta, \dot{\theta}) = \tau_\theta \quad (28)$$

Table 1: Relevant numerical parameters for the four-bar linkage under consideration (given by Wang et al. [1])

Link Lengths (m)	$l_0 = 3.0, l_1 = 1.0, l_2 = 4.0, l_3 = 2.5$
Distance of mass centers (m)	$l_{ci} = l_i/2, i = 1, 2, 3$
Link masses (kg)	$m_i = 1.0, i = 1, 2, 3$
Moment of inertias (kgm ²)	$I_i = m_i l_i^2 / 12, i = 1, 2, 3$
Initial configuration (rad)	$\theta(0) = 1.5708, \alpha(0) = 0.3533, \phi(0) = 1.2649$
Initial velocity (rad/s)	$\dot{\theta}(0) = \dot{\alpha}(0) = \dot{\phi}(0) = 0$
Torque input (N-m)	$\tau_\theta = 6.0, \tau_\alpha = \tau_\phi = 0.0$
Gravitational acceleration (m ² /s)	$g = 9.8$

where τ_θ is the torque applied at the joint θ , and the terms $M(\theta)$ and $V(\theta, \dot{\theta})$ are defined accordingly in Eq. (27), and

$$\begin{aligned}
\frac{\partial S_1}{\partial \theta} &= -\frac{l_1 \cos(\phi - \theta)}{l_2 \sin(\alpha - \phi)} \\
\frac{\partial S_1}{\partial \alpha} &= -\frac{l_1 \sin(\phi - \theta) \cos(\alpha - \phi)}{l_2 \sin^2(\alpha - \phi)} \\
\frac{\partial S_1}{\partial \phi} &= -2 \frac{l_1 \sin(\alpha - \theta)}{-l_2 + l_2 \cos(2\alpha - 2\phi)} \\
\frac{\partial S_2}{\partial \theta} &= -\frac{l_1 \cos(\alpha - \theta)}{l_3 \sin(\alpha - \phi)} \\
\frac{\partial S_2}{\partial \alpha} &= 2 \frac{l_1 \sin(\phi - \theta)}{-l_3 + l_3 \cos(2\alpha - 2\phi)} \\
\frac{\partial S_2}{\partial \phi} &= \frac{l_1 \sin(\alpha - \theta) \cos(\alpha - \phi)}{l_3 \sin^2(\alpha - \phi)} \\
\frac{\partial C_1}{\partial \theta} &= \sin(\alpha - \theta) \\
\frac{\partial C_1}{\partial \alpha} &= -\sin(\alpha - \theta) \\
\frac{\partial G}{\partial \theta} &= -(m_1 l_{c1} + m_2 l_1) g \cos \theta \\
\frac{\partial G}{\partial \alpha} &= -m_2 g l_{c2} \cos \alpha \\
\frac{\partial G}{\partial \phi} &= -m_3 g l_{c3} \cos \phi
\end{aligned}$$

5 Forward Dynamics Simulation

We perform a simple example of forward dynamic problem using the equation of motion derived in the previous section. The numerical parameters for the four-bar mechanism under consideration are listed in Table 1, which is taken from Wang et al. [1]. In the forward dynamic problem, an input torque profile (τ_θ in this case) is actuating the system, and the task is to compute the complete system configuration due to this input. Specifically in this case, we apply a constant torque of $\tau_\theta = 6.0$ N-m at the joint θ .

The mechanism system is simulated using the Real-Time Windows Target¹ in MATLAB/

¹Real-Time Windows Target is a product by MathWorks

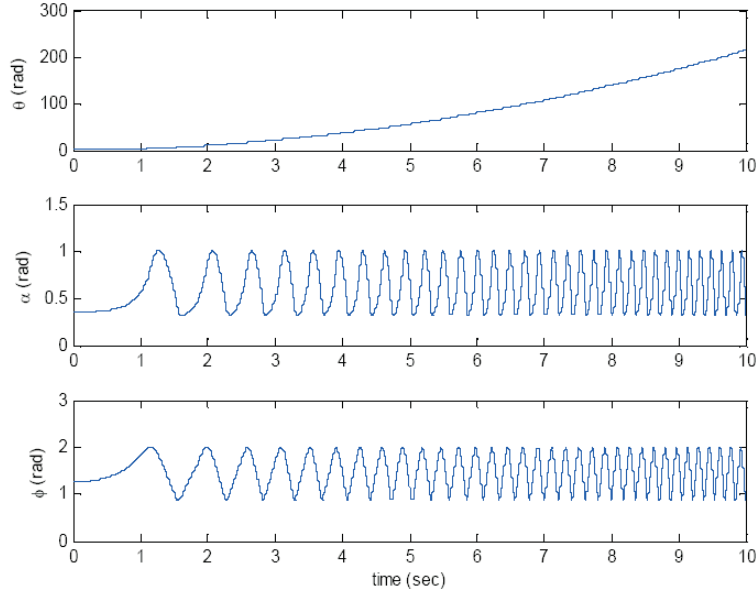


Figure 2: The time evolution of the three angles (θ , α , and ϕ) for the dynamics of the four-bar mechanism under consideration in 10 s

Simulink. The fixed time-step solver of ODE5 scheme (Dormand-Prince) is employed to integrate the equation of motion. Specifically, we use a fixed time-step of $1e-4$ s and simulate the system for a 10s duration, and the position-level time-histories of the three-joints are then plotted in Figure 2. We see that the constant torque input causes acceleration within the system which is reflected in the graphs.

6 Conclusion

In conclusion, we performed the position and velocity analysis for a simple four-bar mechanism problem. Since the system has only 1 d.o.f., we show that the dynamic of the system can be completely parameterized by only one coordinates (in this case, the input angle θ .) Finally, a simple forward dynamic simulation was performed in the real-time setting for completeness.

7 Acknowledgment

Special thanks to Dr. D. B. Doman for pointing out the typos of the earlier version of this report.

References

- [1] J. Wang, C.M. Gosselin, and L. Cheng, “Modeling and simulation of robotic systems with closed kinematic chains using the virtual spring approach,” *Multibody System Dynamics*, vol. 7, no. 2, pp. 145–170, 2002.